ON A PROBLEM OF OPTIMUM CONTROL OF NONLINEAR SYSTEMS

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Consider the system of differential equations

$$\frac{dx}{dt} = f(x,\ t) + q(t)\eta(t) \tag{0.1}$$

where $f(s,\ t)$ and $q(t)$ are known functions of their arguments. In what follows small roman letters (with the exception of the letter t which denotes the time and n the order of the system) will denote n-dimensional vectors, small greek letters scalars, and capital roman letters square matrices of order n. The symbol $\|y\|$ will denote the norm $\|y\| = (y_1^2 + \ldots + y_n^2)^{1/2}$ of the vector y. We will assume that the functions f are defined and continuously differentiable for $t \geqslant t_0$ at all points of the space x, except at the points lying on the surfaces

$$\xi_\alpha(x,\ t) = 0 \quad (\alpha = 1, \ldots, \mu) \tag{0.2}$$

of the space $x \times t$. The surfaces (0.2) do not intersect, and in the neighborhoods of these surfaces the functions ξ_α are supposed to be continuously differentiable. The functions $q(t)$ are piecewise smooth and have (if any) only a finite number of discontinuities of the first kind in every bounded interval $t_0 \leqslant t \leqslant T$ *.

Under the condition

$$|\eta(t)| \leqslant 1 \tag{0.3}$$

let the problem of optimum regulation of fast operation for the system (0.1) be formulated [1-3, 4], i.e. it is required to determine a piecewise smooth function $\eta^0(t)$ such that for a given initial instant $t = t_0$, a point $x = x_0$ and a smooth curve $x = z(t)$ the moving point $x(x_0,\ t_0,\ t,\ \eta^0)$

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* The capital letter T will also denote the time.

of the trajectory of system (0.1), where $\eta(t) = \eta^0(t)$, reaches the curve $x = z(t)$ in the shortest possible time $T^0 = t - t_0$. Obviously, without loss of generality, we can assume that $t_0 = 0$ and $z(t) \equiv 0$, since in the contrary case it is sufficient only to carry out the transformation of the time $\tau = t - t_0$ and of the coordinates $y = x - z(t)$. Henceforward, therefore, we will assume that $t_0 = 0$, $z(t) \equiv 0$.

In the papers [1-3] a maximum principle for the solution of such problems in the most general case of smooth stationary functions $f(x, u)$ (u being a ρ-dimensional control vector) is proposed and justified.

In conformity with the approach to the problems of optimum regulation as described in paper [4], in the present paper certain existence problems as well as necessary and sufficient criteria for the existence of optimum trajectories of a nonstationary system (0.1) with discontinuities (0.2) are considered.

The argument is carried out for the general case of systems of order n. However, for $n > 2$ an efficient formulation of the theorems is difficult. The difficulty in passing from systems of order two to systems of order n originates in the fact that for nonlinear systems in the case $n > 2$ it is in general impossible to give an efficient rule for the verification of the condition of complete linear independence of the resolving functions $h(T, \tau)$ (see below p. 211). In particular we will note all cases for which the assertions are correct only when $n = 2$.

1. Let us introduce certain definitions.

1. The surface $\xi_\alpha(x, t) = 0$ is a section for the trajectory $x(x_0, t, \eta)$* if in a neighborhood of the point of intersection $x(x_0, t_\alpha, \eta)$ with the surface $\xi_\alpha = 0$ we have $\xi_\alpha < 0$ for $t < t_\alpha$, $\xi_\alpha > 0$ for $t > t_\alpha$ and the inequalities

$$\lim_{t\to t_\alpha+0}\Big(\sum_\beta \frac{\partial \xi_\alpha}{\partial x_\beta} f_\beta + \frac{\partial \xi_\alpha}{\partial t}\Big) = \frac{d\xi_\alpha}{dt^+} > \varepsilon > 0 \tag{1.1}$$

and

$$\lim_{t\to t_\alpha-0}\Big(\sum_\beta \frac{\partial \xi_\alpha}{\partial x_\beta} f_\beta + \frac{\partial \xi_\alpha}{\partial t}\Big) = \frac{d\xi_\alpha}{dt^-} > \varepsilon > 0 \tag{1.2}$$

are satisfied.

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* In conformity with the choice $t_0 = 0$ the letter t_0 in $x(x_0, t_0, t, \eta)$ is omitted, denoting the trajectory of (0.1) by the symbol $x(x_0, t, \eta)$.

Here it is assumed that on the surfaces (0.2) the functions f and their derivatives have discontinuities of the first kind.

2. Assume that in the interval $0 \leqslant t \leqslant T$ the trajectory $x(x_0, t, \eta)$ intersects the surfaces (0.2) at the points $t = t_a (a = 1, \ldots, \mu)$. Moreover, we will assume that these surfaces are numbered in the order of increasing t_a. Let us assign to the function $\eta(t)$ the variation $\delta\eta(t)$ and let us construct the system of linear equations for the variations along the trajectory $x(x_0, t, \eta)$. If for the trajectory under consideration the surfaces (0.2) are sections, then in constructing the variational equations it is necessary to follow the rule justified in the papers [5,6], i.e. the system of linear approximation for the perturbations due to the variations $\delta\eta(t)$ include linear differential equations

$$\frac{d\delta x}{dt} = P(t)\,\delta x + q(t)\,\delta\eta(t) \qquad (1.3)$$

where the elements of the matrix $P(t)$ are to be calculated by the formulas

$$\{P\}_{\alpha\beta} = \left(\frac{\partial f_\alpha}{\partial x_\beta}\right)_{x=x(x_0,\, t,\, \eta)} \qquad \text{for } t \neq t_\alpha \qquad (1.4)$$

and the linear discontinuities of the quantities δx, when passing through the points $t = t_a$, are determined by the relations

$$\delta x(t_\alpha + 0) = A(t_\alpha)\,\delta x(t_\alpha - 0) \qquad (1.5)$$

The coefficients of the matrix $A(t_a)$ are to be calculated by the formulas [5]

$$\{A(t_\alpha)\}_{\beta\gamma} = \delta_{\beta\gamma} + \Delta_\beta(\alpha)\,\zeta_\gamma^{-\alpha} \qquad \left(\zeta_\gamma^{\alpha-} = \frac{\partial\xi_\beta / \partial x_\gamma}{d\xi_\alpha / dt^-}\right) \qquad (1.6)$$

where $\delta_{\beta\beta} = 1$, $\delta_{\beta\gamma} = 0$ for $\beta \neq \gamma$, and $\Delta_\beta(a)$ denotes the magnitude of the discontinuity f_β at the point $x = x(x_0, t_a, \eta)$.

Applying impulse functions, linear discontinuities (1.5) can be included into the system (1.3). This, however, does not essentially simplify the argument.

The system, consisting of equations (1.3) to (1.5), will be called the variational system.

3. The trajectory $x(x_0, t, \eta)$ of system (0.1) will be called admissible if the function $\eta(t)$ satisfies the condition (0.3) and the surfaces (0.2) intersected by this trajectory are sections for the latter.

4. To the variational system corresponding to a certain admissible trajectory $x(x_0, t, \eta)$, let us apply the Cauchy formula for the solution of nonhomogeneous linear systems [7] (p. 172). Putting the initial variations $\delta x(0)$ equal to zero and taking into account the rules (1.5) we can

write

$$\delta x(t) = B_1 \int_0^{t_1} F_0(t_1) F_0^{-1}(\tau) q(\tau) \delta\eta(\tau) d\tau + \tag{1.7}$$

$$+ \sum_{\alpha=1}^{\mu-1} B_{\alpha+1} \int_{t_\alpha}^{t_{\alpha+1}} F_\alpha(t_{\alpha+1}) F_\alpha^{-1}(\tau) q(\tau) \delta\eta(\tau) d\tau + \int_{t_\mu}^{t} F_\mu(t) F_\mu^{-1}(\tau) q(\tau) \delta\eta(\tau) d\tau$$

where $F_a(t)$ is a fundamental matrix of the solutions of system (1.3) for $t_a \leqslant t < t_{a+1}$,

$$F_\alpha(t_\alpha) = E, \qquad B_\alpha = F_\mu(t) A(t_\mu) F_{\mu-1}(t_\mu) \ldots F_\alpha(t_{\alpha+1}) A(t_\alpha)$$

and μ denotes the number of surfaces (0.2) intersected by the trajectory $x(x_0, t, \eta)$.

If the right-hand side of (1.7) is written down in the form of a single integral, the quantities $\delta x(t)$ are determined by the formula

$$\delta x(t) = \int_0^t h(t, \tau) \delta\eta(\tau) d\tau. \tag{1.8}$$

where the vector $h(t, \tau)$ is expressible in a well-known manner in terms of the functions which are on the right-hand side of (1.7). In what follows the solutions of the variational system will be written at once in the form (1.8).

The function $h(t, \tau)$ will be called the resolving vector of the variational system. In what follows the vector function $h(t, \tau)$ will also be written in the form $h(t, \tau) = D(t, \tau)q(\tau)$ or (for $t_a < \tau < t_{a+1}$) in the form $h(t, \tau) = G(t, a)F_a^{-1}(\tau)q(\tau)$, where the matrices $D(t, \tau)$ and $G(t, a)$ can be expressed in a well-known manner in terms of the functions which are on the right-hand side of (1.7).

5. The resolving vector $h(t, \tau)$ will be called nonsingular if the scalar product $(h(t, \tau).l)$ of the vector h by an arbitrary nonzero vector l can vanish only for separate isolated values of τ (for fixed t).

Let us note here a criterion which allows us to decide when for a nonlinear system of the second order the resolving vector $h(t, \tau)$ is nonsingular*.

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* As the dimension n increases, these criteria become extremely complicated. Therefore we will not quote them in their general form. Later in Section 2 we will quote a criterion for a vector $h(t, \tau)$ to be nonsingular in the case $n = 3$.

Lemma 1.1. Let $Q(x, t)$ be matrix determined by the equality

$$\{Q(x, t)\}_{\alpha\beta} = \frac{\partial f_\alpha(x, t)}{\partial x_\beta} \qquad (\alpha, \beta = 1, \ldots, n) \tag{1.9}$$

where, at the points on the surfaces (0.2), the limit values Q^+ and Q^- are to be taken according as the approach to the surfaces takes place from the domain $\xi_\alpha > 0$ or $\xi_\alpha < 0$, respectively. If for all x from the domain G of the space $\{x\}$ and for $0 \leqslant t \leqslant T$ the vector $q(t)$ is not collinear with the vector $dq/dt - Q(x, t)q(t)$ (including the limit values of Q, q, dq/dt at the points of discontinuities), then the resolving vector $h(T, t)$ of the variational system, calculated along an arbitrary admissible trajectory $x(x_0, t, \eta)$ and lying for $0 \leqslant t \leqslant T$ in the domain G, is nonsingular.

Proof. For $t = t^*$ let the equality

$$(h(T, t^*)\cdot l) = 0 \qquad \text{for} \|l\| \neq 0 \tag{1.10}$$

be satisfied. It $t = t^*$ is a point of discontinuity of $Q(x(t), t)$ or $q(t)$, then $h(T, t^*)$ stands for the right- or left-hand limit of $h(T, t)$. For reasons of definiteness assume that in the equality (1.10) $h(T, t^*)$ denotes the right-hand value of $h(T, t)$, and let us calculate the right-hand derivative $d(h(T, t).l)/dt^*$. By formula (1.7) we have

$$\frac{d(h(T, t)\cdot l)}{dt^+} = \left(\left[G(T, \alpha)\frac{d(F_\alpha^{-1}(t)q(t))}{dt^+}\right]\cdot l\right) \qquad \text{for } t \geqslant t_\alpha,\ t < t_{\alpha+1} \tag{1.11}$$

where $G(T, a)$ is a certain nonsingular matrix which can be expressed in a well-known manner in terms of the matrices F_γ and B_β (see p. 211). It is well-known [7] that (p. 171)

$$\frac{dF_\alpha^{-1}(t)}{dt} = -F_\alpha^{-1}(t)P(t)$$

and therefore it follows from (1.11) that

$$\frac{d(h(T, t)\cdot l)}{dt^+} = \left(\left[G(T, \alpha)F_\alpha^{-1}(t)\left(\frac{dq}{dt^+} - P(t)q(t)\right)\cdot l\right)\right. \qquad (P(t) = Q(x(x_0, t, \eta), t))$$

According to the assumptions of the lemma the vector $dh/dt^+ = GF_\alpha^{-1}(t)(dq/dt^+ - Pq)$ is colinear with the vector $h = GF_\alpha^{-1}(t)q$ and, consequently, the two equalities

$$h\cdot l = 0, \qquad dh/dt^+\cdot l = 0 \qquad \text{for } \|l\| \neq 0$$

cannot be satisfied simultaneously. Consequently, at the point $t = t^*$ we have

$$\frac{d(h(T, t)\cdot l)}{dt^+} \neq 0$$

i.e. $(h.l) \neq 0$ in a neighborhood of the point $t = t^*$ and to the right of it (for $t > t^*$).

Similarly, it is proved that from the condition $(h.l) = 0$ for the left-hand limit $h(T, t^*)$ follows $(h.l) \neq 0$ for small $t - t^* < 0$. Hence the Lemma is proved.

Notes. 1. If the vector $q(t)$ is piecewise constant, then $dq/dt^{\pm} = 0$ outside the points of discontinuity. Therefore from Lemma 1.1 it follows that, in order that the resolving vector $h(T, t)$ is nonsingular, it is sufficient that the vector $q(t)$ is not a characteristic vector of the matrix $Q(x, t)$. If the domain G is bounded and the inequalities (1.1) and (1.2) are satisfied uniformly for all admissible curves, then in the domain G for $t \in [0, T]$, $(h.l) = 0$ and $\|l\| = 1$, the quantity $|d(h(T,t).l)/dt^{\pm}|$ has a positive minimum. In this case, under the assumptions of Lemma 1.1, the resolving vector $h(T, t)$ is nonsingular in a stronger sense. Namely, there exists a constant number $\gamma > 0$ such that the measure of the set Σ_γ on $[0, T]$, where

$$(h(T, t)\cdot l) \leqslant \delta \tag{1.12}$$

satisfies the inequality

$$\operatorname{mes} \Sigma_\delta \leqslant \gamma\delta \tag{1.13}$$

no matter what is the vector l ($\|l\| = 1$) and the admissible curve $x(x_0, t, \eta)$ along which the system of variational equations is calculated.

2. Along with the vector $h(T, t)$ consider also the vector

$$g(t) = C(t_\alpha)F_\alpha^{-1}(t)\,q(t) \qquad \text{for } t_\alpha < t < t_{\alpha+1}$$

where

$$C(t_\alpha) = F_0^{-1}(t_1)A^{-1}(t_1)\ \ldots A^{-1}(t_{\alpha-1})F_{\alpha-1}^{-1}(t_\alpha)$$

Since for $0 \leqslant t \leqslant T$ the vector functions $h(T, t)$ and $g(t)$ are connected by a linear nonsingular transformation $g(t) = H.h(T, t)$, the conditions of nonsingularity of the vectors $h(T, t)$ and $g(t)$ coincide. Therefore the conditions of nonsingularity for $h(T, t)$, which were proved above and will be deduced below, are also the conditions of nonsingularity for $g(t)$. Later on we shall not make particular mention of this fact. In what follows functions of the type $\eta(t) = \operatorname{sign}(h(T, t).l)$ will be considered, where l is a certain nonzero vector and $0 \leqslant t \leqslant T$. The same function can also be determined by the formula $\eta(t) = \operatorname{sign}(g(t).l')$, where the vector l' is connected with the vector l by means of a well-known nonsingular linear transformation.

6. Let us quote from the book [8] a result which will be of essential use in what follows.

Consider the problem of determining a function $\zeta(t, c)$, satisfying the condition

$$|\zeta(t,\ c)|\leqslant 1 \qquad (0\leqslant t\leqslant T) \tag{1.14}$$

and being such that the equality

$$c=\int_0^T h(T,\ \tau)\zeta(\tau,\ c)\,d\tau \tag{1.15}$$

holds.

According to a theorem from [8] (pp. 171-179) the function $\zeta(t,\ c)$ for the given vector c, the time T and the resolving vector $h(T,\ t)$ exists if and only if the inequality

$$\min_{(l\cdot c)=1}\left(\int_0^T |(h(T,\tau)\cdot l)|\,d\tau\right)\geqslant 1 \tag{1.16}$$

is satisfied.

If the vector $h(T,\ t)$ is nonsingular and in the condition (1.16) the equality sign holds, there exists a unique solution of the problem (1.15) (to within values on a set of measure zero which we neglect). The function $\zeta(t,\ c)$ is defined by the formula

$$\zeta(t,\ c)=\operatorname{sign}(h(T,\ t)\cdot l^{\circ}) \tag{1.17}$$

l° being the vector which resolves the problem (1.16).

In the paper [9] it is shown that the functions $\zeta(t,\ c)$, which solve the problem (1.15), can be selected in such a way that they are continuous in measure with respect to the vector c, i.e. the function $\zeta(t,\ c)$ will converge in measure to the function $\zeta(t,\ c^*)$ as $c\to c^*$.

In the vector space $\{c\}$ the set of points c for which, for a given T and a resolving vector $h(T,\ t)$, the problem (1.15), (1.14) is solvable, is a closed and convex set containing the point $c = 0$ [8] (pp. 171-179). For the given T and h this set will be called the domain of attainability and will be denoted by $\Lambda\ (T,\ h)$. From the results of the book [8] it also follows that in the case of nonsingular vectors h the domain $\Delta\ (T,\ t)\to \Lambda\ (T,\ h^*)$ whenever $T\to T^*$ and $h(T,\ t)\to h^*(T^*,\ t)$ in measure on the segment $[0,\ T^*]$.

2. For a piecewise smooth system (0.1) the necessary conditions of optimum control, briefly presented for smooth functions f and in detail for nonstationary linear systems in an earlier paper [4] are justified in this section. By means of more complicated proofs, the arguments given below can be extended to the case of several control functions $\eta_1(t)$, $\ldots,\ \eta_p(t)$, each being subjected to the condition (0.3). In this article, however, we shall restrict ourselves to the case of one control function $\eta(t)$.

Let the system possess an optimum trajectory $x(x_0\ \ t\ \ \eta^0)$, T^0 be the optimum time of control, and $\eta^0(t)$ the optimum control function itself, determining the optimum trajectory under consideration.

In conformity with the plan outlined in paper [4], to deduce the necessary conditions of optimum control it is necessary to consider the quantity

$$\sigma[\eta] = \sup|\eta(t)| \qquad \text{for } 0 \leqslant t \leqslant T^\circ \tag{2.1}$$

strictly speaking, it is necessary to consider the quantity

$$\sigma^*[\eta] = vrai\sup|\eta(t)| \ \text{ for } 0 \leqslant t \leqslant T^\circ \tag{2.2}$$

i.e. the upper bound of the quantity $|\eta(t)|$ on the segment $[0,\ T^0]$, excepting subsets of measure zero. This circumstance, however, will be neglected in what follows without justifying it specifically. Therefore, instead of (2.2) we will consider the quantity (2.1)]. We will assume the fulfilment of the following conditions:

1. The optimum trajectory $x(x_0,\ t,\ \eta^0)$ connects the points $x = x_0$ and $x = 0$, and intersects μ hypersurfaces (0.2) for $t = t_a < T^0$ $(a = 1, \ldots, \mu)$.

2. For the variational system, consisting of equations (1.3) and (1.5) and calculated along the trajectory $x(x_0,\ t,\ \eta^0)$, the resolving vector $h^0(T^0,\ t)$ is nonsingular.

Theorem 2.1. Let the assumptions 1 and 2 be satisfied. Then on the optimum control function $\eta^0(t)$ the functional $\sigma[\eta]$ assumes a relative minimum

$$\sigma[\eta^\circ] = \sigma_{\min} = 1 \tag{2.3}$$

for the variations $\delta\eta(t)$ restricted by the condition

$$\int_0^{T^\circ} h^\circ(T^\circ,\ \tau)\,\delta\eta(\tau)\,d\tau = 0 \tag{2.4}$$

Proof. Assume the contrary, i.e. that for the optimum control frunction $\eta^0(t)$ the conditions (2.3) and (2.4) are not satisfied. First let

$$\sigma[\eta^\circ] = \alpha < 1 \tag{2.5}$$

be satisfied. Then under conditions (2.5) and the result mentioned in part 6 of Section 1, for an arbitrary vector c, such that

$$\|c\| \leqslant \delta \qquad (\delta > 0) \tag{2.6}$$

and for sufficiently small $\delta > 0$, there exists a function $\zeta_\nu(t,\ c)$ which satisfies the conditions

$$\int_0^{T_\nu} h^\circ(T_\nu,\ \tau)\,\zeta_\nu(\tau,\ c)\,d\tau = c \qquad (\nu = 1,\ 2,\ \ldots) \tag{2.7}$$

and

$$|\zeta_\nu(t,\ c)| \leqslant 1 - \bar{\alpha} \text{ при } 0 \leqslant t \leqslant T_\nu = T^\circ - \tau_\nu \tag{2.8}$$

where $\{r_\nu\}$ is a certain monotone null sequence of positive numbers, $r_1 < 1/2\, T^0$. For every fixed value $\nu \geqslant 1$, according to the results of [9] mentioned in part 6 of Section 1, the functions $\zeta_\nu(t,\ c)$ can be selected in such a way that they are continuous in the measure of c, taken from (2.6). Let us define the variations $\delta\eta_\nu(t,\ c,\ \mu)$ of the functions $\eta^0(t)$ by the formulas

$$\delta\eta_\nu(t,\ c,\ \mu) = \mu\zeta_\nu(t,\ c) \tag{2.9}$$

where $\mu > 0$ is a small parameter. Consider the solutions of the variational system $\delta x^{(\nu)}(t,\ c,\ \mu)$, corresponding to the variations (2.9).

By formula (1.8), as a consequence of (2.7) and (2.9), we have

$$\delta x^{(\nu)}(T_\nu,\ c,\ \mu) = \int_0^{T_\nu} h^\circ(T_\nu,\ \tau)\,\delta\eta_\nu(\tau,\ c,\ \mu)\,d\tau = \mu c \tag{2.10}$$

From equality (2.10), according to the definition (2.9) of the variations $\delta\eta_\nu$, we conclude that the endpoint of the vector $y = \delta x^{(\nu)}(T_\nu,\ c,\ \mu)$ for every $\nu \geqslant 1$ describes a sphere

$$\|y\| \leqslant \mu\delta \tag{2.11}$$

whenever the vector c runs through the domain (2.6). In addition, because of (2.5) and (2.8) the inequality

$$|\eta^\circ(t) + \delta\eta_\nu(t,\ c,\ \mu)| \leqslant 1 \qquad \text{for } 0 \leqslant t \leqslant T_\nu,\ \mu \leqslant 1 \tag{2.12}$$

is satisfied.

Now consider the behavior of the trajectories of system (0.1) corresponding to the control functions $\eta(t) = \eta^0(t) + \delta\eta_\nu(t,\ c,\ \mu)$.

The deviations $\delta^\nu x^{(\nu)}(t,\ c,\ \mu)$ of these trajectories from the considered optimum trajectory $x(x_0,\ t,\ \eta^0)$ will satisfy the complete equations for the perturbed motion

$$\frac{d\delta^\nu x}{dt} = P(t)\delta^\nu x + q(t)\,\delta\eta_\nu(t,\ c,\ \mu) + r(\delta^\nu x,\ t) \tag{2.13}$$

which, in contrast to the variational system consisting of (1.3) and (1.5), contain on the right-hand side additional functions $r(\delta^\nu x,\ t)$. In every interval of continuity $(t_\alpha + \beta,\ t_{\alpha+1} - \beta)$ $(\beta > 0)$ the functions $r(\delta^\nu x, t)$ are small and of higher order with respect to δx, i.e.

$$r(\delta^\nu x,\ t) = o(\|\delta^\nu x\|) \tag{2.14}$$

In the neighborhoods of the points of discontinuity $t = t_\alpha$ the functions $r(\delta^\nu x,\ t)$ contain, besides the terms of the order of (2.14), terms which are compensated for, up to within the order of (2.14), by linear jumps (1.5) of the solutions of the variational system. Taking into account (2.9) and repeating the arguments of the papers [5,6] it can be verified that for the variations $\delta\eta_\nu(t, c, \mu)$ the actual deviations $\delta^\nu x^{(\nu)}(t, c, \mu)$ differ from the solutions $\delta x^{(\nu)}(t, c, \mu)$ of the variational system by a quantity of the order higher than μ, i.e.

$$\|\delta^\nu x^{(\nu)}(t, c, \mu) - \delta x^{(\nu)}(t, c, \mu)\| = o(\mu) \tag{2.15}$$

We will not here give the derivation of estimation (2.15), since it can be obtained by methods well-known in the qualitative theory of differential equations and in the theory of stability [7] (pp. 19-22). Let us further remark only that the estimation (2.15) is determined by properties of continuity of the functions f, $\partial f_\gamma/\partial x_\beta$, $\partial f_\gamma/\partial t$, $\partial\xi_\alpha/\partial x_\beta$, $\partial\xi_\alpha/\partial t$ (in the domains of their continuity) and the values of T^0 and ϵ from (1.1) and (1.2). Therefore, if instead of a single unperturbed trajectory $x(x_0, t, \eta^0)$, as hitherto, but a set of such trajectories is considered, as will be done incidentally in the sections which follow, and if the estimations of the continuity and the numbers T^0 and ϵ are uniform with respect to the whole set of unperturbed trajectories, then the estimation (2.15) will also be satisfied uniformly, i.e.

$$\|\delta^\nu x^{(\nu)}(t, c, \mu) - \delta x^{(\nu)}(t, c, \mu)\| \leqslant \varphi(\mu)\mu, \quad \varphi(\mu) \to 0 \qquad \text{for } \mu \to 0 \tag{2.16}$$

will hold good for the whole set of trajectories.

Let the number $\mu_0 > 0$ be selected in such a way that the condition

$$\|\delta^\nu x^{(\nu)}(t, c, \mu) - \delta x^{(\nu)}(t, c, \mu)\| \leqslant \tfrac{1}{16}\delta\mu_0 \qquad \text{for } 0 \leqslant t \leqslant T_\nu \tag{2.17}$$

holds good.

Then from (2.11) and the inequality (2.17) we conclude that the points $x(x_0, T_\nu, \eta^0 + \delta\eta_\nu(t, c, \mu_0)$ for every $\nu \geqslant 1$ fill up a certain manifold $\Sigma(\nu)$ which embraces the point $x(x_0, T_\nu, \eta^0)$. Since for every fixed $\nu \geqslant 1$ the variations $\delta\eta_\nu(t, c, \mu_0)$ change continuously in measure with the variation of c in (2.6), the manifold $\Sigma(\nu)$ is a continuous image of the sphere

$$\|x(x_0, T_\nu, \eta^\circ) - x\| \leqslant \mu_0\delta \tag{2.18}$$

and the points $x \in \Sigma(\nu)$, being images of the points, lying on the boundary of the sphere (2.18), are at a distance less than $1/10\ \delta\mu_0$ from this boundary. Owing to the fact that, for sufficiently large values of $\nu, \tau_\nu \to 0$, the inequality

$$\|x(x_0, T_\nu, \eta^\circ)\| \leqslant \tfrac{1}{16}\mu_0\delta \tag{2.19}$$

is satisfied, and therefore for such values of ν the manifold $\Sigma(\nu)$ will embrace the point $x = 0$. Here the manifld $\Sigma(\nu)$ can be considered as a continuous image of the sphere

$$\|x\| \leqslant \mu_0 \delta \tag{2.20}$$

such that the points of $\Sigma(\nu)$, being the images of the points lying on the surface of the sphere (2.20), are at a distance less than $1/8\ \mu_0\delta$ from the boundary of this sphere.

On the basis of a theorem on the roots [10] we can conclude that there exists a variation $\delta\eta_{\nu_0}(t, c^*, \mu_0)$ for which the equality

$$x(x_0, T_{\nu_0}, \eta^\circ + \delta\eta_{\nu_0}(t, c^*, \mu_0)) = 0 \tag{2.21}$$

is satisfied. This means that if the condition (2.12) is satisfied the trajectory $x(x_0, t, \eta^0 + \delta\eta_{\nu_0}(t, c^*, \mu_0))$ will reach the point $x = 0$ in the time $t = T_{\nu_0} < T^0$ (for a sufficiently large and fixed value of $\nu = \nu_0$), i.e. the considered original trajectory $x(x_0, t, \eta^0)$ is not an optimum trajectory. The contradiction obtained shows that for the optimum control the inequality (2.5) is impossible.

Now assume that the equality

$$\sigma[\eta^\circ] = 1 \tag{2.22}$$

is satisfied but the quantity $\sigma[\eta^0]$ is not minimal with respect to the perturbations of the control function $\delta\eta(t)$ restricted by the condition (2.4). Then there exists a variation $\mu\delta\eta(t)$ such that

$$\sigma[\eta^\circ + \mu\delta\eta] = \sup|\eta^\circ(t) + \mu\delta\eta(t)| = 1 - \varepsilon\mu \qquad (0 \leqslant \mu \leqslant 1) \tag{2.23}$$

holds.

Here $\epsilon = \text{const} > 0$, $\mu > 0$ is a parameter and

$$\int_0^{T^\circ} h^\circ(T^\circ, \tau)\,\mu\delta\eta(\tau)\,d\tau = 0 \tag{2.24}$$

Owing to the condition (2.24), the solutions $\delta x(t, \mu)$ of the variational system, and the corresponding variations $\mu\delta\eta(t)$, satisfy the condition

$$\delta x(T^\circ, \mu) = 0$$

Since for $t = T^0$ the optimum trajectory $x(x_0, t, \eta^0)$ reaches the point $x = 0$, and the deviation of the actual trajectory of (0.1), corresponding to the variation $\mu\delta\eta(t)$, from the solution of the variational system satisfies the estimation (2.15), then

$$\|x(x_0, T^\circ, \eta^\circ + \mu\delta\eta)\| = o(\mu) \tag{2.25}$$

holds good.

On the other hand the difference $1 - \sigma[\eta + \mu\delta\eta] = \epsilon\mu$ is small and of the first order with respect to μ. Therefore, repeating the arguments used at the beginning of the proof, we arrive at the conclusion that for the sequence $T_\nu = T^0 - r_\nu (r_\nu > 0)$ it is possible to indicate a small number μ_0 and a variation $\delta\eta\,(t,\ c,\ \mu_0)$ of the function $\eta(t) = \eta^0(t) + \mu_0\delta\eta\,(t)$ such that the solutions $\delta\, x^\nu(t,\ c,\ \mu_0)$ of the variational system fill up the sphere (2.11) (for $\mu = \mu_0$ and a certain constant $\delta > 0$), and the deviations of the linear approximation $\delta x^{(\nu)}$ from the solutions of the complete equations $\delta^\nu x^{(\nu)}$ are less than $1/16\ \delta\mu_0$. Since the radius of the sphere (2.11) decreases linearly with variation of μ, and the distance (2.25) is of higher order of smallness with respect ot μ, the number μ_0 can be selected so small that the condition

$$\| x\,(x_0, T^\circ, \eta_i^\circ + \mu_0\,\delta\eta_i) \| < \tfrac{1}{16}\,\delta\mu_0$$

holds.

If the number $r_\nu = T^0 - T_\nu$ is so small that the inequality

$$\| x\,(x_0, T_\nu, \eta_i^\circ + \mu_0\delta\eta_i) - x\,(x_0, T^\circ, \eta_i^\circ + \mu_0\,\delta\eta_i) \| < \tfrac{1}{16}\,\mu_0\delta$$

is satisfied, then it can be asserted that the points

$$x = x\,(x_0, T_\nu, \eta_0 + \mu_0\delta\eta_i + \delta\eta_i\,(t, c, \mu_0))$$

fill up a manifold $\Sigma(\nu)$, when the endpoint of the vector c runs through the sphere (2.6), where $\Sigma(\nu)$ is a continuous image of the sphere (2.20). The points $x \in \Sigma(\nu)$, being images of the points lying on the boundary of the sphere (2.20), are from the points of this boundary at a distance less than $3/16\ \mu_0\delta$.

Now, as before, according to a theorem on the roots [10], we conclude that there exists a trajectory $x(x_0,\ t,\ \eta^0 + \delta\eta\,(t,\ c^*,\ \mu_0))$ arriving at the point $x = 0$ for $t = T_{\nu_0} < T^0$, where the control function $\eta(t) = \eta^0(t) + \mu_0\delta\eta + \delta\eta(t,\ c^*,\ \mu_0)$ satisfies the condition (0.3). This fact again contradicts the assumption that $x(x_0,\ t,\ \eta^0)$ is the optimum trajectory.

The contradictions arrived at prove the theorem.

The theorem proved allows us to establish the form of the optimum function $\eta^0(t)$. Namely, the following assertion holds.

Theorem 2.2. If under the assumptions 1 and 2 $\eta^0(t)$ is the optimum trajectory $x(x_0,\ t,\ \eta^0)$, then

$$\eta_i^\circ(t) = \operatorname{sign}\,(h^\circ(T^\circ, t)\cdot l^\circ) \tag{2.26}$$

where l^0 is a certain nonzero vector [1].

---

* According to the remark 2 of Section 1 on p. 9 the optimum control function $\eta^0(t)$ is also determined by the formula $\eta^0(t) = \operatorname{sign}\ (g(t).\ l')$, where l' is a certain nonzero vector.

Proof. Consider the vector

$$c^\circ = \int_0^{T^\circ} h^\circ(T^\circ, \tau)\,\eta^\circ(\tau)\,d\tau \qquad (2.27)$$

The assertion of Theorem 2.1 means that there does not exist a function $\eta(t)$ satisfying the conditions

$$c^\circ = \int_0^{T^\circ} h^\circ(T^\circ, \tau)\,\eta(\tau)\,d\tau \qquad (2.28)$$

and

$$\sup|\eta(t)| = \alpha < 1 \qquad \text{for } 0 \leqslant t \leqslant T^\circ \qquad (2.29)$$

In conformity with the results of paper [8], mentioned in part 6 of Section 1, the function $\eta^0(t)$ satisfying these conditions is determined uniquely and has the form (2.26). This proves the Theorem 2.2.

Note. According to the results of paper [8] (see Section 1) the vector l^0 satisfies the condition

$$1 = \min_{(c^\circ\cdot l)=1} \int_0^{T^\circ} |(h^\circ(T^\circ, \tau)\cdot l^\circ|\,d\tau \qquad (2.30)$$

In the case of nonlinear systems it is most difficult to make use of conditions (2.26) and (2.30) for an effective determination of the optimum control since neither the vector c^0 nor the resolving vector function $h^0(T^0, t)$ are known in advance (in the case of a linear system the resolving function $h(T, t)$ is one and the same for all trajectories (for a given T), and c^0 is the vector $F(T)x_0$). In the nonlinear case the function $h(T, t)$ depends on the trajectory which in turn is determined by the control function $\eta(t)$. The basic difficulty consists in the choice of such a vector l^0 for which the corresponding optimum trajectory arrives at the prescribed finite point $x = 0$. As a rule, for the determination of the vector l^0 it is impossible to obtain equations which can be effectively solved, since the equations (0.1) in the majority of cases cannot be integrated in an elementary form for $\eta^0(t)$ given by (2.26). This difficulty can be avoided by selecting the vector l^0 on a trial basis. In such a case it is necessary to pass from the system of differential equations to the corresponding system of difference equations with a small step Δt:

$$\Delta x^{(\rho)} = [f(x(\rho\Delta t), \rho\Delta t) + q(\rho\Delta t)\,\eta(\rho\Delta t)]\,\Delta t \qquad (\rho = 0, 1, \ldots) \qquad (2.31)$$

The connection between the problem of optimum control for difference and linear differential equations is discussed in the paper [11]. The difference equations (2.31) can be integrated stepwise as follows. Let us give the numbers l_β^0, being the projections of l^0, and let us calculate at the initial point the values $h_\beta^0(T^0, 0)$ [actually, at the point $x = x_0$, $\rho = 0$ only the quantities $g_\beta(0)$, being projections of $g(t)$, can be calculated. These are obtained from $h_\beta^0(T^0, 0)$ by a nonsingular linear transformation (see the remark 2 of Section 1 on p. 9). As a consequence of an arbitrary selection of l^0 this circumstance is inessential, and in formula (2.26) we can write $g(t)$ instead of $h^0(T^0, t)$]. Given l^0 and $g(0)$, determine $\eta^0(0)$ from the condition (2.26). Afterwards, given $\eta^0(0)$, determine $x(\Delta t) = x_0 - \Delta x^{(0)}$ from the system (2.31) for $\rho = 0$. Now determine $g(\Delta t)$ at the point $x = x(\Delta t)$, and again $\eta(\Delta t)$ by the formula (2.26), and so on*.

If after a sufficient number of steps the trajectory determined in this way does not come close to the point $x = 0$, another vector l^0 must be tested, and so on. Since there is no definite rule for the determination of l^0, the conditions which are given by the stated necessary criterion must be considered mainly as guiding ideas for the determination of the optimum trajectory.

Let us note a corollary of Theorems 2.1 and 2.2 in the case of second order systems.

Corollary 2.1. Let for all x from the domain G and for $0 \leqslant t \leqslant T^0$ the vector $q(t)$ be not colinear with the vector $dq/dt - Qq$, where the matrix Q is determined by the equality (1.9). If $x(x_0, t, \eta^0)$ is the optimum trajectory and the hypersurfaces of discontinuities (0.2) intersected by it are sections for it, then the optimum control function $\eta^0(t)$ is a piecewise smooth function of the form $\eta^0(t) = \text{sign}\,(h^0(T^0, t). l^0)$ [or $\eta^0(t) = \text{sign}\,(g(t). l')$]. For $0 \leqslant t \leqslant T^0$ the quantity $\sigma[\eta] = \sup|\eta(t)|$ assumes on this function a relative minimum

$$\sigma[\eta^\circ] = \sigma_{min} = 1 \tag{2.32}$$

for variations $\delta\eta$ restricted by the condition**

$$\int_0^{T^\circ} h^\circ(T^\circ, \tau)\,\delta\eta(\tau)\,d\tau = 0 \tag{2.33}$$

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* For the determination of $g(\rho\Delta t)$ $(\rho = 0, 1, \dots)$ it is necessary to calculate the fundamental matrix of the solutions of the variational system consisting of equations (1.3) to (1.5), which also can be replaced by finite difference equations.

** Or, what is the same, by the condition

$$\int_0^{T^\circ} g(\tau)\,\delta\eta(\tau)\,d\tau = 0$$

In particular, if the functions f are smooth and the vector $q(t)$ is not colinear with the vector $dq/dt - Qq$, the conditions (2.32) and (2.33) are necessary for a trajectory to be an optimum trajectory.

The truth of Corollary 2.1 follows at once from Lemma 1.1 and Theorems 2.1 and 2.2.

Now consider the system (0.1) for $n = 3$. Let the functions $f(x, t)$ on the right-hand sides of (0.1) be continuous and have continuous second order partial derivatives with respect to all arguments while the vector q is constant. Let us prove first sufficient conditions for which the resolving vector $h(T^0, t)$ is nonsingular.

Lemma 2.1. For the variational system (1.3) constructed along an arbitrary admissible trajectory $x(x_0, t, \eta)$, lying for $0 \leqslant t \leqslant T$ in the domain G, the vectors $h(T, t)$ will be nonsingular provided that the following conditions are satisfied: the vectors q, Qq and Rq are not coplanar for $x \in G$, $0 \leqslant t \leqslant T$ and $-1 \leqslant \eta \leqslant 1$. Here the matrix Q is determined by (1.9) while the matrix R is defined by

$$\{R\}_{ij} = \sum_{\alpha=1}^{n} \frac{\partial f_i}{\partial x_\alpha} \frac{\partial f_\alpha}{\partial x_j} - \sum_{\alpha=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_\alpha} (f_\alpha(x, t) + q_\alpha \eta) - \frac{\partial^2 f_i}{\partial x_j \partial t} \qquad (2.34)$$

To prove the lemma it is sufficient to note that the following equalities hold (the verification of which is omitted):

$$\frac{d(h(T^\circ, t) \cdot l)}{dt} = ([D(T^\circ, t) Qq] \cdot l), \qquad \frac{d^2(h(T^\circ, t) \cdot l)}{dt^2} = ([D(T^\circ, t) Rq] \cdot l) \qquad (2.35)$$

where D is a nonsingular matrix such that $h(T^0, t) = D(T^0, t)q$ (see p. 211). Since the vector q, Qq and Rq are not coplanar, there does not exist a vector $l \neq 0$ satisfying the conditions

$$(h \cdot l) = 0, \ \left(\frac{dh}{dt} \cdot l\right) = 0, \quad \left(\frac{d^2 h}{dt^2} \cdot l\right) = 0$$

i.e. the function $(h(T^0, t).l)$ can vanish only at separate isolated points of t, i.e. the vector $h(T^0, t)$ is nonsingular.

From Lemma 2.1 and Theorems 2.1 and 2.2 follows the following assertion.

Corollary 2.1. If for $0 \leqslant t \leqslant T^0$ the trajectory $x(x_0, t, \eta^0)$, connecting the points $x = x_0$ and $x = 0$, is the optimum trajectory and lies in the domain G, where for $0 \leqslant t \leqslant tT^0$, $-1 \leqslant \eta \leqslant 1$ the vectors q, Qq and Rq are not coplanar, then the optimum control function $\eta^0(t)$ generating this trajectory, is a piecewise smooth function $\eta^0(t) = \operatorname{sign}\,(h^0(T^0, t).l^0)$ (or else $\eta^0(t) = \operatorname{sign}\,(g(t).l'))$. Here $l^0(l')$ is a certain nonzero constant vector, the function $\eta^0(t)$ is a solution of the problem for $0 \leqslant t \leqslant T^0$, min max $|\eta(t)| = 1$ for the variations $\delta\eta$ restricted by the conditions

$$\int_0^{T^\circ} h^\circ(T^\circ, \tau)\, \delta\eta(\tau)\, d\tau = 0$$

where $h^0(T, t)$ is the resolving vector for the variational system (1.3) constructed along $x = x(x_0, t, \eta^0)$.

3. In this section we will clarify certain problems concerning the existence of optimum trajectory with a piecewise smooth control function, taking into account the nonlinearity of system (0.1). We will assume that the finite point $x = 0$ does not lie on a hypersurface of the family (0.2) and that the functions f and q on the right-hand side of system (0.1) satisfy the conditions

$$\| f(x, t) \| \leqslant \lambda_1 \| x \| + \lambda_2, \qquad \| q(t) \| \leqslant \lambda_3 \qquad (\lambda_{1,\ldots,3} = \text{const}) \tag{3.1}$$

for all x and $t \in [0, T]$. The conditions (3.1) are satisfied in any case if the functions f possess, in the domains of their continuity, uniformly bounded partial derivatives $\delta f_\beta/\delta x_\gamma$. For further investigations the conditions (3.1) are not necessary but they make the discussions simpler.

Let there exist at least one control function $\eta(t)$ which satisfies condition (0.3) and is such that the corresponding trajectory $x(x_0, t, \eta)$ connects the initial point $x = x_0$ with the final point by an arc $0 \leqslant t \leqslant T$.

Let $T_\nu (\nu = 1, 2, \ldots)$ be a monotone nonincreasing sequence of numbers such that for every $\nu \geqslant 1$ there exists a control function $\eta_\nu(t)$ which satisfies the condition (0.3) and is such that the corresponding trajectory $x(x_0, t, \eta)$ satisfies the condition

$$x(x_0, T_\nu, \eta_\nu) = 0 \tag{3.2}$$

Moreover, assume that there does not exist a control function $\eta^*(t)$ which satisfies (0.3) and

$$x(x_0, t, \eta^*) = 0 \tag{3.3}$$

for $t < T_\infty$, where $T_\infty = \lim T_\nu$ as $\nu \to \infty$.

If for a certain $\nu \geqslant 1$ the equality $T_\nu = T_\infty$ is satisfied, then, assuming that the trajectory $x(x_0, t, \eta_\nu)$ intersects the hypersurfaces $\xi_a = 0$ and satisfies the section conditions (1.1) and (1.2) while the resolving vector $h^{(\nu)}(T_\nu, t)$ of the corresponding variational system is nonsingular, it can be asserted that as a consequence of (3.3) the trajectory $x(x_0, t, \eta_\nu)$ is the optimum trajectory and according to Theorems 2.1 and 2.2 there corresponds to this trajectory a piecewise smooth control function $\eta_\nu(t)$ of the form $\eta_\nu(t) = \operatorname{sign}(h^\nu(T_\nu, t).l)$.

Therefore it is of interest to consider only the case when for every $\nu \geqslant 1$ the inequality

$$T_\nu > T_\infty \qquad (\nu = 1, 2, \ldots) \tag{3.4}$$

is satisfied; and, consequently, there exists an infinite set of trajectories $x^{(\nu)}(t) = x(x_0, t, \eta_\nu)$, forming a minimizing sequence $\{x^{(\nu)}(t)\}$.

Let us make the following remark. The functions f on the right-hand sides of system (0.1) may have discontinuities on the hypersurfaces $\xi_\alpha = 0$. We will assume that the curves $x^{(\nu)}(t)$, beginning with a sufficiently large number ν, intersect the hypersurfaces $\xi_\alpha = 0$ and satisfy the section conditions (1.1) and (1.2).

We will restrict the class of admissible control functions $\eta(t)$ to piecewise smooth functions only. If we admit a larger class of functions for which there exist solutions of the system (0.1) in a generalized sense defined in the paper [12], then the proof can be simplified. Here in our argument we will confine ourselves to classical solutions only, although this lengthens the proof somewhat.

We will prove that under certain restrictions such a piecewise smooth control function $\eta^0(t)$ actually exists and determines the optimum trajectory $x(x_0, t, \eta^0)$, this being the limit trajectory for the subsequence $\{x^{(\nu)}(t)\}$, while the function $\eta^0(t)$ itself is the function to which the functions $\eta_\nu(t)$ converge in measure on the segment $[0, T_\infty]$. We will here give a short outline of the proof. A detailed proof for the existence of the optimum control function in the case of smooth functions has been given by Kirillova.

First, using condition (3.1), by arguments typical for problems on continuation of trajectories [7] (pp. 17-19), it can be verified that the family of functions $x^{(\nu)}(t)$ is uniformly bounded for $0 \leqslant t \leqslant T_\infty$, and therefore as a consequence of (3.1) this family is also equi-continuous. Therefore from the sequence $\{x^{(\nu)}(t)\}$ we can extract a uniformly convergent subsequence $\{x^{(\nu\beta)}(t)\}$ $(\beta = 1, 2, \ldots)$.

Assume that the limit function $x^\infty(t)$ satisfies conditions (1.1) and (1.2) and that the resolving vector $h^\infty(T_\infty, t)$ of the variational system constructed formally along the curve $x = x^\infty(t)$ is nonsingular.

The further problem is to prove that the curve $x = x^\infty(t)$ is the optimum trajectory for the system (0.1) and that this trajectory corresponds to a piecewise smooth control function $\eta^0(t)$. Let us prove this. The subsequence $\{x^{(\nu\beta)}(t)\}$ will be reenumerated by the numbers $\nu = 1, 2, \ldots$

Let $h^{(\nu)}(T_\infty, t)$ be the resolving vector functions of the variational systems calculated along the trajectories $x^{(\nu)}(t)$. Since for $\nu \to \infty$ on the intervals of continuity $t_a^\infty - \gamma < t < t_a^\infty + \gamma$ ($\gamma > 0$) the matrices of the coefficients of these systems $P^{(\nu)}(t)$ converge uniformly to the matrix $P^\infty(t)$ of the coefficients of the limit variational system, evaluated along $x = x^\infty(t)$, the numbers $t_a^{(\nu)}$ converge to t_a^∞ ($t = t_a$ are the instants of intersection of the trajectories with the hypersurfaces $\xi_a = 0$), then the resolving vector functions $h^{(\nu)}(T_\infty, t)$ converge in measure to the vector function $h^\infty(T_\infty, t)$.

Consider the sequence of vectors

$$c^{(\nu)} = \int_0^{T_\infty} h^\infty(T_\infty, \tau)\,\eta_\nu(\tau)\,d\tau \tag{3.5}$$

Since $|\eta_\nu(t)| = 1$, all the points $c^{(\nu)}$ lie in the domain of attainability $\Lambda(T_\infty, h^\infty)$ (see pp.213). The points $c^{(\nu)}$ as $\nu \to \infty$ converge to a certain set of points, lying on the boundary of $\Lambda(T_\infty, h^\infty)$. If this were not the case, then, repeating with inessential modifications the arguments of the proof of Theorem 2.1) (pp. 214-217), we could construct a trajectory $x(x_0, t, \eta)$ arriving at the point $x = 0$ for $t = T_\infty - \tau^*$, where $\tau^* > 0$. This, however, is impossible.

Now consider a certain convergent subsequence $c^{(\nu_\gamma)}$. Let $\lim c^{(\nu_\gamma)} = c^\infty$, where $c = c^\infty$ lies on the boundary of $\Lambda(T_\infty, h^\infty)$ and $\eta^\infty(\tau)$ is a piecewise smooth function which satisfies the conditions

$$c^\infty = \int_0^{T_\infty} h^\infty(T_\infty, \tau)\,\eta^\infty(\tau)\,d\tau \tag{3.6}$$

and

$$|\eta^\infty(t)| \leqslant 1 \qquad \text{for } 0 \leqslant t \leqslant T_\infty \tag{3.7}$$

Since c^∞ lies on the boundary of $\Lambda(T_\infty, h^\infty)$, then in conformity with part 6 of Section 1 and by virtue of certain theorems from paper [8], the function $\eta^\infty(t)$ is determined uniquely by the condition

$$\eta^\infty(t) = \operatorname{sign}(h^\infty(T_\infty, t)\cdot l^\infty) \qquad (l^\infty \neq 0,\ l^\infty = \text{const}) \tag{3.8}$$

Let us prove that the function $\eta_{\nu_\gamma}(t)$ converges in measure to $\eta^\infty(t)$. In fact, the sequence of vectors

$$e^{(\nu_\gamma)} = c^{(\nu_\gamma)} - c^\infty \qquad (\gamma = 1, 2, \ldots) \tag{3.9}$$

satisfy the conditions

$$\lim e^{(\nu_\gamma)} = 0 \qquad \text{for } \gamma \to \infty \tag{3.10}$$

and

$$e^{(\nu_\gamma)} = \int_0^{T_\infty} h^\infty(T_\infty, \tau)\,\zeta^{(\nu_\gamma)}(\tau)\,d\tau \qquad (\zeta^{(\nu_\gamma)}(\tau) = \eta_{\nu_\gamma}(\tau) - \eta^\infty(\tau)) \tag{3.11}$$

Multiplying scalarly the left-hand side of (3.11) by l^{∞}, we obtain

$$(e^{(\nu_\gamma)}\cdot l^{\infty}) = \int_0^{T_\infty} (h^{\infty}(T_\infty, \tau)\cdot l^{\infty})\,\zeta^{(\nu_\gamma)}(\tau)\,d\tau \tag{3.12}$$

As a consequence of the equality $\eta^{\infty}(t) = \text{sign}\,(h^{\infty}(T_\infty, t).\,l_\infty)$ the function under the integral sign on the right-hand side of (3.12) does not change its sign (the sign of $\zeta^{(\nu_\gamma)}(t)$, obviously, can be only opposite to that of $\eta^{\infty}(t)$). Because of (3.10) the left-hand side of (3.12)tends to zero as $\gamma \to \infty$. As a consequence of the remark just made about the conservation that $(h^{\infty}.\,l^{\infty})$ can vanish only on a set of measure zero, it follows that the functions $\zeta^{(\nu_\gamma)}(t)$ tend in measure to zero on the segment $[0,\ T_\infty]$. Thus the functions $\eta_{\nu_\gamma}(t)$ actually converge in measure to $\eta^{\infty}(t)$.

Now it is not difficult to verify that after the substitution $\eta^0(t) = \eta^{\infty}(t)$ in the right-hand side of (0.1) the corresponding solution $x(x_0,\ t, \eta^0)$ on the segment $[0,\ T_\infty]$ coincides with the function $x = x^{\infty}(t)$, since otherwise the functions $x^{(\nu)}(t)$ and $\eta_{\nu_\gamma}(t)$ could not simultaneously converge uniformly to $x^{\infty}(t)$ could not simultaneously converge uniformly to $x^{\infty}(t)$, and in measure to $\eta^{\infty}(t)$, respectively.

If we assume that the sequence $c^{(\nu)}$ has at least two different limit points (c^{∞}) and $(c^{\infty})^*$, then two control functions $(\eta^{\infty}(t))$ and $(\eta^{\infty}(t))^*$ should exist which are different on a set of measure zero and such that the corresponding trajectories $x(x_0,\ t,(\eta^{\infty}))$ and $x(x_0,\ t,(\eta^{\infty}))^*$ coincide for $t\in[0,\ T_\infty]$. This, however, is impossible.

The contradiction obtained shows that the sequence $\eta_\nu(t)$ converges in measure to a uniquely determined piecewise smooth function $\eta^{\infty}(t) = \eta^0(t)$ which at the same time is the optimum control function. Hence the assertion is proved.

Now consider a second-order system (0.1).

Let the system (0.1) be smooth and let the assumptions of Lemma 1.1 be satisfied in the shole space, i.e. 1) for all x and $0 \leqslant t \leqslant T$ the vectors $q(t)$ and $dq/dt - Q(x,\ t)\,q(t)$ are not colinear. Then as a corollary of the results obtained above and Lemma 1.1 we obtain the following assertion.

Corollary 3.1. If a second-order system satisfies the assumptions 1) and (3.1), and there exists at least one trajectory $x(x_0,\ t,\ \eta)$ connecting the points $x = x_0$ and $x = 0$ by an arc $0 \leqslant t \leqslant T$, where the control function $\eta(t)$ satisfies the condition (0.3), then the system (0.1) possesses an optimum piecewise smooth control function $\eta^0(t)$ of the form $\eta^0(t) = \text{sign}\,\lambda(t)$, where the function $\lambda(t)$ may vanish only at separate points of $t\in[0,\ T]$.

If the nonlinear system (0.1) is of higher order than two and the trajectory $x = x(t)$ along which the variational system is calculated is not known in advance, an effective verification of the fulfilment of the conditions of nonsingularity of the resolving vector $h(T, t)$ becomes very difficult.* If, however, for the variational system, calculated along a known admissible trajectory $x = x(x_0, t, \eta)$, the verification of the nonsingularity of the vector $h(T, t)$ is considered, this nonsingularity can be verified without solving the variational system and without determining the vector $h(T, t)$ itself, But also in this case with the increase of n, the sufficient conditions of the nonsingularity of $h(T, t)$ become very rapidly more and more complicated. Let us here quote one sufficient condition of the nonsingularity of h in the case of a quasi-linear system, assuming that the vector q is constant.

In the case of a linear system all variational systems calculated for various trajectories coincide, and therefore the conditions of the nonsingularity of the resolving vector $h(T, r)$ can be formulated effectively without knowing the trajectory in advance. Let us here quote, for example, such conditions for a third-order system under the assumption that the vector q is constant and the functions $P(t)$ in the linear system under consideration

$$\frac{dx}{dt} = P(t)x + q\eta(t) \tag{3.13}$$

are piecewise smooth with discontinuities (if any) of the first kind only.

Lemma 3.1. If for $t \in [0, T]$ the vectors q, Pq and $p^2 q - (dP/dt)q$ do not lie in a linear two-dimensional space, then the resolving vector $h(T, t)$ of the system (3.13) is nonsingular.

The truth of Lemma 3.1 follows from Lemma 2.1 since in the case of a linear system along any trajectory $x = (t)$ the equalities $Q = P(t)$ and $R = p^2 - dP/dt$ are satisfied.

Calculating successively higher order derivatives of $(h. l)$, analogous conditions for the nonsingularity of the resolving vector $h(T, t)$ can be derived for the linear system (3.13) and for the general case n. Since these conditions have a cumbersome form, we shall not give them here.

Consider a quasilinear system (0.1), i.e. a system of the form

$$\frac{dx}{dt} = P(t)x + \mu r(x, t) + q\eta(t) \tag{3.14}$$

* See, for example, Lemma 2.1 in which are stated the sufficient conditions for nonsingularity of h when $n = 3$.

where the functions $r(x, t)$ possess continuous and uniformly bounded partial derivatives $\partial r_\beta/\partial x_\alpha$, $\partial r_\beta/\partial t$, $\partial^2 r_\beta/\partial x_\alpha \partial x_\gamma$, $\partial^2 r_\beta/\partial x_\alpha \partial t$. Since for small μ the vectors Qq and Rq (see Lemma 2.1) differ slightly from the vectors Pq and $(P^2 - dP/dt)q$, then from Lemmas 2.1 and 3.1 we obtain as a corollary the following result.

Corollary 3.2. If q is a constant vector and the vectors q, Pq and $(P^2 - dP/dt)q$ do not lie for every t from $0 \leqslant t \leqslant T$ in a linear two-dimensional subspace, then for sufficiently small μ the resolving vector $h(T, t)$ of the variational system, calculated along any admissible curve $x = x(x_0, t, \eta)$, is nonsingular.

From Corollary 3.2 follows the validity of the following conclusion.

Corollary 3.3. Let the parameter μ in the right-hand side of the system (3.14) be selected sufficiently small, and suppose that the constant vector q and the vectors Pq and $P^2 q - (dP/dt)q$ do not lie in a linear two-dimensional space for every $t \in [0, T]$. If there exists at least one admissible curve $x(x_0, t, \eta)$, connecting the points $x = x_0$ and $x = 0$ by an arc $x(x_0, t, \eta)$ $(0 \leqslant t \leqslant T)$, the system (3.14) possesses a piecewise smooth optimum control function $\eta^0(t)$ of the form $\eta^0(t) = \operatorname{sign} \lambda(t)$, to which corresponds the optimum trajectory $x(x_0, t, \eta^0)$.

Let us here quote without proof a corollary which can easily be derived from the results of Section 3.

Corollary 3.4. Consider the system of equations

$$\frac{dx}{dt} = f(x) + bu$$

and denote by A the matrix

$$\{A\}_{ij} = \left(\frac{\partial f_i}{\partial x_j}\right)_{x=0}$$

If the vectors b, Ab, $\ldots$, $A^{n-1}b$ are linearly independent, there exists a neighborhood of the point $x = 0$ such that for an arbitrary point $x = x_0$ there exists an optimum trajectory $x(x_0, t, \eta^0)$, connecting the points $x = x_0$ and $x = 0$, the optimum control function $\eta^0(t)$ having the form $\eta^0(t) = \operatorname{sign} \lambda(t)$, where the function $\lambda(t)$ vanishes only at separate isolated points t.

4. In this section certain sufficient conditions for optimum control will be proved. We will consider systems of order n, assuming that the only hypersurfaces of discontinuity $\xi_\alpha = 0$ are the hypersurfaces $t = t_\alpha =$ const. Among the admissible curves the optimum trajectory $x(x_0, t, \eta)$, according to the results of Section 2, is that one for which the control function $\eta^0(t)$ has the form

$$\eta^\circ(t) = \mathrm{sign}\,(h^\circ(T^\circ, t)\cdot l^\circ) \tag{4.1}$$

where l^0 is a certain constant nonzero vector.

Let us prove that under certain restrictions conditions (4.1) are also necessary for the existence of a local optimum trajectory of the system (0.1). The latter term is to be understood in the following sense.

Definition 4.1. The trajectory $x^0(t) = x(x_0, t, \eta^0)$, connecting the points $x = x_0$ and $x = 0$ by the arc $0 \leqslant t \leqslant T^0$, where the control function $\eta^0(t)$ satisfies the condition (0.3), will be said to be a local optimum trajectory if there exists a number $\epsilon > 0$ such that no motion $x(t) = x(x_0, t, \eta)$ exist which satisfies the system of equations (0.1) with the control function $\eta(t)$ subject to condition (0.3) and which connects the points $x = x_0$ and $x = 0$ by the arc $0 \leqslant t \leqslant T$ with $T < T^0$ and is such that

$$\| x(x_0, t, \eta^\circ) - x(x_0, t, \eta) \| < \varepsilon \qquad \text{for } 0 \leqslant t \leqslant T$$

holds.

In other words, the trajectory $x^0(t)$ must be an optimum trajectory with respect to arbitrary variations $\delta\eta$ restricted by condition (0.3) and with respect to sufficiently small displacements of the trajectory itself.

Theorem 4.1. Let the following assumptions be satisfied:

1. The trajectory connects the points $x = x_0$ and $x = 0$ by the arc $0 \leqslant t \leqslant T^0$, i.e.

$$x(x_0, 0, \eta^\circ) = x_0 \tag{4.2}$$

$$x(x_0, T^\circ, \eta^\circ) = 0 \tag{4.3}$$

2. The control function $\eta^0(t)$ is of the form (4.1), where $h^0(T^0, t)$ is the resolving vector of the variational system calculated along $x(x_0, t, \eta^0)$.

3. For the vector l^0, determining the control function $\eta^0(t)$, the property of being nonsingular is satisfied in a stronger sense, namely the measure of the set Σ_δ, where the inequality

$$|(l^\circ\cdot h^\circ(T^\circ, t))| \leqslant \delta \tag{4.4}$$

holds, satisfies the inequality

$$\mathrm{mes}\,(\Sigma_\delta) \leqslant \gamma\delta \qquad (\gamma = \mathrm{const}) \tag{4.5}$$

Obviously, for conditions (4.4) and (4.5) to be satisfied, it is sufficient, at the points for which $(l^0.h^0) = 0$, for the left- and right-hand derivatives of $(l^0.h^0)$ to satisfy the conditions

$$\frac{d(l^\circ \cdot h^\circ)}{dt^+} \neq 0, \quad \frac{d(l^\circ \cdot h^\circ)}{dt^-} \neq 0 \tag{4.6}$$

4 The vector e which is tangential to the trajectory $x(x_0, t, \eta^0)$ at the point $t = T - 0$ and is in the direction of increasing t, satisfies the condition*

$$(e \cdot l^\circ) > 0 \tag{4.7}$$

5. Denote by $a(t_k)$ the vectors defined by the equalities

$$a(t_k) = \int_{t_k}^{T^\circ} F(T^\circ, t_k) F^{-1}(\tau, t_k) \Big[\sum_{i=1,\, j=1}^{n} \frac{\partial^2 f(x^\circ(\tau), \tau)}{\partial x_i \partial x_j} h_i(\tau, t_k) h_j(\tau, t_k) \Big] d\tau$$

where $F(t, t_k)$ is a fundamental matrix of solutions of the variational system

$$\frac{d\delta x}{dt} = P(t)\,\delta x$$

$(F(t_k, t_k)$, $h_a(t, \tau)$ is the component of the resolving vector of this system and t_k the time instants for which the equalities

$$(l^\circ \cdot h^\circ(T^\circ, t_k)) = 0$$

are satisfied.

We will assume that the inequalities $(a_k . e) > 0$ are satisfied.

Under these assumptions the trajectory $x = x(x_0, t, \eta^0)$ is a local optimum trajectory.

Note. If the same control function $\eta^0(t)$ can be determined by formula (4.1) by means of various vectors l^0, it is sufficient for condition (4.7) to be satisfied for at least one such vector l^0. In the formulation of Theorem 4.1 it is naturally assumed that the trajectory $x(x_0, t, \eta^0)$ does not intersect the points $x = 0$ for $t = T_1 < T^0$.

Condition 5 can be replaced by the requirement that the second partial derivatives $\partial^2 f_a / \partial x_\beta \partial x_\gamma$ shall be sufficiently small.

Let us note also that by means of an example it can be shown that in

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* Obviously the vector e is defined by the equality

$$e = \lim\,[f(x(x_0, t, \eta^\circ), t) + q\eta^\circ(t)] \qquad \text{при } t \to T^\circ - 0$$

provided that this limit is different from zero.

the general case the conditions 1-4 are not sufficient for $x(x_0, t, \eta^0)$ to be a local optimum trajectory.

Proof. Assume the contrary, i.e. that there exists a sequence of numbers $\epsilon_\nu > 0$ ($\nu = 1, 2, \ldots$), converging to zero, and a corresponding sequence of control functions $\eta_\nu(t)$, satisfying condition (0.3) and such that the trajectories $x(x_0, t, \eta_\nu)$ satisfy the conditions

$$x(x_0, T^\circ - \tau_\nu, \eta_\nu) = 0 \text{ for } \tau_\nu > 0 \qquad (\nu = 1, 2, \ldots) \tag{4.8}$$

and

$$\| x(x_0, t, \eta_\nu) - x(x_0, t, \eta^\circ) \| < \varepsilon_\nu \qquad (\nu = 1, 2, \ldots) \tag{4.9}$$

Let $\zeta_\nu = \eta_\nu - \eta^0$. Since the functions η_ν and η^0 satisfy the condition (0.3), and the function $\eta^0(t)$ is defined by the formula (4.1), the sign of $\zeta_\nu(t)$ is always opposite to that of $(h^0(T^0, t).l^0)$. Let us calculate the scalar products of the vectors $\delta x^{(\nu)}(t)$, which are solutions of the system of variational equations

$$\delta x^{(\nu)}(T_\nu) = \int_0^{T_\nu} h^\circ(T_\nu, \tau)\, \delta\eta_\nu(\tau)\, d\tau \qquad (\delta\eta_\nu(t) = \zeta_\nu(t), T_\nu = T^\circ - \tau_\nu) \tag{4.10}$$

with the vectors $l^{(\nu)} = [\,(F(T_\nu).F^{-1}(T^0)^*]^{-1}\, l^0$. We have*

$$(\delta x^{(\nu)}(T_\nu) \cdot l^{(\nu)}) = \int_0^{T_\nu} (h^\circ(T_\nu, \tau) \cdot l^{(\nu)})\, \delta\eta_\nu(\tau)\, d\tau \tag{4.11}$$

From conditions (4.4) and (4.5) we conclude that the functions $\delta\eta_\nu(t)$ converge in measure to zero as $\nu \to \infty$. The deviations $\delta^\nu x^{(\nu)}$ of the actual trajectories of (0.1) from the trajectory $x(x_0, t, \eta^0)$, caused by the variations $\delta\eta_\nu$ from the corresponding solutions of the variational system, are of the order $o(\epsilon_\nu)$. For $\nu \to \infty$ the left-hand sides of the equalities (4.11) must converge to zero. On the other hand, if the measure of the set $(\Sigma_\delta^{(\nu)})$ under condition

$$|\delta\eta_\nu| > \delta \qquad \text{for } 0 \leqslant t \leqslant T_\nu \tag{4.12}$$

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* The sign * in the formula for $l^{(\nu)}$ means the transpose of a matrix and $F(t)$ a fundamental matrix of solutions of the variational system (1.3). As a consequence of $\epsilon_\nu \to 0$ we have $\tau_\nu \to 0$ for $\nu \to \infty$. Therefore by the definition of the vectors $l^{(\nu)}$ and the vector $h^0(t, \tau)$ (see p. 211) we have, obviously, the equality $(h^0(T^0, t).l^0) - (h^0(T_\nu, t).l^{(\nu)})$, and, in addition, $l^{(\nu)} \to l^0$ as $\nu \to \infty$. Without loss of generality we shall assume that $||l^0|| = 1$.

is greater than $\alpha > 0$, then according to condition 3 of the theorem we have

$$|(l^{(\nu)} \cdot \delta x^{(\nu)}(T_\nu))| \geqslant \gamma_1 \delta\alpha \qquad (\gamma_1 = \mathrm{const}) \tag{4.13}$$

From the inequality (4.13) we conclude that for $\nu \to \infty$ the functions converge in measure to zero. Denote

$$\mu_\nu = \int_0^{T_\nu} |\delta\eta_\nu(t)|\, dt \tag{4.14}$$

Hence $\lim \mu_\nu = 0$ as $\nu \to \infty$.

First consider the case when from the sequence μ_ν a subsequence can be extracted (in order to simplify notations let us identify this subsequence with the original sequence μ_ν) for which the conditions

$$\left| \int_0^{T^\circ - \tau_\nu} (h(T^\circ - \tau_\nu, \tau) \cdot l^\circ)\, \delta\eta_\nu(\tau)\, d\tau \right| > \beta\mu_\nu \tag{4.15}$$

are satisfied and where β is a fixed, sufficiently small, positive number. The deviations of the actual solutions $\delta^\nu x$ of the system (0.1) from the linear approximation δx, i.e. $\delta^\nu x = x(x_0, t, \eta^0 + \delta\eta_\nu) - x(x_0, t, \eta^0)$ caused by the variations $\delta\eta_\nu$, will be of the order $o(\mu_\nu)$. This fact can be easily verified by the usual arguments of the qualitative theory. However, we will not carry out this verification here. At the same time by virtue of condition 3 of the theorem and the above coincidence of the signs of $\delta\eta_\nu(t)$ and $(h^0(T_\nu, t) . l^{(\nu)})$ the quantities (4.11) are small and of the first order with respect to μ_ν, and negative for $\nu = 1, 2, \ldots$.

Thus the vectors $\delta^\nu x^{(\nu)}(T_\nu)$ will also be small and of the first order with respect to μ_ν. In addition, these vectors must converge to the vector e since, by the definition of $\delta x^{(\nu)}$ and the trajectories $x(x_0, t, \eta^0 + \delta\eta_\nu)$, we have

$$\check{\delta} x^{(\nu)}(T_\nu) = -x(x_0, T_\nu, \eta^\circ)$$

The scalar products $(\delta x^{(\nu)} . l^{(\nu)})$ and $(\delta^\nu x^{(\nu)} . l^{(\nu)})$, as it was proved, differ by an infinitely small quantity of higher order, and consequently, the quantities $(\delta^\nu x^{(\nu)} . l^{(\nu)})$ must be small and of the first order with respect to μ_ν and negative for large values of ν. This contradicts the convergence of $l^{(\nu)} \to l^0$ in direction and condition 4 of Theorem 4.1.

Now consider the second possible case when

$$\left| \int_0^{T_\nu} (h(T_\nu, \tau) \cdot l^\circ)\, \delta\eta_\nu(\tau)\, d\tau \right| < q_\nu \mu_\nu$$

where $q_\nu \to 0$ as $\nu \to \infty$. In this case it cannot be asserted that the variations $\delta x^\nu(T_\nu)$ are small and of the first order with respect to μ_ν. Therefore second-order terms in μ_ν must be taken into account. These additional second order terms, as it can be verified by solving the complete equations of the perturbed motion by successive approximations, will be equal to the vectors determined by condition 5 of the theorem and multiplied by certain positive quantities $\beta_{\nu k}$.

Denote by $\delta_2 x^\nu(t)$ the second approximation of the solution $\delta^\nu x^\nu(t)$.

Now, as in the first case considered, we arrive at the contradiction, namely the scalar products $(\delta x_2{}^{(\nu)} . l^{(\nu)})$ and $(\delta^\nu x^{(\nu)} . l^{(\nu)})$, which differ from each other by infinitely small quantities of higher order than $\mu_\nu{}^2$, must be negative for large values of ν. This fact, however, contradicts the conditions 4 and 5 of the theorem.

The obtained contradictions prove the theorem.

Notes. *1*. In the conditions of the theorem it was assumed that the hypersurfaces of discontinuities $\xi_a = 0$ are the planes $t = t_a$. This assumption was needed in order to make use of the estimation $\| \delta^\nu x - \delta x \| = o(\mu_\nu)$ in the process of the proof, the derivation of which without the assumption that these hypersurfaces are sections for the trajectories under consideration, would be in general impossible (in any case on the basis of the nongeneralized solutions $x(t)$ of system (0.1)). In consequence of the boundedness of the right-hand sides of the system (0.1) along the trajectories $x(x_0, t, \eta)$ which differ from the trajectory $x(x_0, t, \eta^0)$ by a small quantity $\epsilon > 0$, the property of being sections for these trajectories is possessed also by the hypersurfaces $\xi_a = 0$, not necessarily coinciding with $t = t_a$, but being such that the normal vector grad ξ_a is inclined to the t-axis at a sufficiently small angle. Therefore the assertion of Theorem 4.1 remains in force in the case of such hyper surfaces of discontinuities also.

2. If we consider only variations which are restricted by the conditions $|\delta\eta_\nu| < \epsilon_\nu$, then from the fact that the hypersurfaces of discontinuities $\xi_a = 0$ are sections for the trajectory $x(x_0, t, \eta^0)$, it will follow that they are sections also for the trajectories $x(x_0, t, \eta^0 + \delta\eta_\nu)$, i.e. to prove that the trajectory $x(x_0, t, \eta^0)$ is a local optimum trajectory in the sense of small variations δx as well as in the sense of small variations of $\delta\eta$, it is sufficient only to assume in Theorem 4.1 that the hypersurfaces $\xi_a = 0$ are sections for the trajectory $x(x_0, t, \eta^0)$.

3. For condition 3 of Theorem 4.1 to be satisfied, it is sufficient that the vectors

$$h^\circ(T^\circ, t),\ h'(T^\circ, t) = D[Pq - q']$$

where the matrix $D(T^0, t)$ determines the vector $h = D(T^0, t)q(t)$ (see p. 211) do not lie simultaneously in the hypersurfaces $(l^0.h^0) = 0$, $(l^0.h') = 0$. In fact, in such a case, repeating the arguments of the proof of Lemma 1.1, we would obtain that

$$\left| \frac{d\,(h^\circ (T^\circ,\ t)\cdot l^\circ)}{dt^\pm} \right| \geqslant \gamma > 0 \qquad (\gamma = \text{const}) \tag{4.16}$$

holds at every point of the trajectory, where $(l^0.h^0) = 0$, i.e. condition 3 of the theorem, in fact, is satisfied.

4. In the case of a quasilinear system (3.14) with sufficiently small μ for condition 3 of Theorem 4.1, in conformity with Corollary 3.3 and the previous note, to be satisfied, it is sufficient for the vectors

$$F\,(T^\circ)\,F^{-1}\,(t)\,q\,(t), \qquad F\,(T^\circ)\,F^{-1}\,(t)\left[P\,(t)\,q\,(t) - \frac{dq}{dt}\right]$$

where $F(t)$ denotes a fundamental matrix of solutions of the linear system, not to lie for $0 \leqslant t \leqslant T^0$ simultaneously in the hypersurface $(l^0.h) = 0$.

5. In the particular case of a second-order system for condition 3 of the theorem to be satisfied, it is sufficient for the vectors q and $Qq - q'$ to be noncolinear.

6. Finally, let us note that conditions 1, 3 and 4 of Theorem 4.1 correspond in the stationary case to the rule for the construction of the optimum trajectory on the basis of the Pontryagin maximum principle [1].

Example. Finally, consider a simple case of a concrete system by means of which we shall illustrate the possibility of verifying all the conditions imposed on the systems considered in proving the theorems of Sections 2-4.

Let the equation

$$\frac{d^2x}{dt^2} + f\,(x,\ t) = \eta\,(t) \tag{4.17}$$

be given, where x and f are scalars, the function $f(x, t)$ is continuously differentiable for $t_\alpha \leqslant t \leqslant t_{\alpha+1}$ and its partial derivatives admit (if any) at the points $t = t_\alpha$ only discontinuities of the first kind. We shall also assume that the following conditions

$$f\,(0,\ t) = 0, \qquad \omega_1 < \frac{\partial f}{\partial x} < \omega_2 \qquad \text{for any } x \qquad (\omega_1 > 0,\ \omega_2 = \text{const} > 0) \tag{4.18}$$

are satisfied.

The system of two equations, equivalent to (4.17), is

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -f\,(x,\ t) + \eta\,(t) \tag{4.19}$$

The variational system corresponding to (4.19) has the form

$$\frac{d\delta x}{dt} = \delta y, \quad \frac{d\delta y}{dt} = -p(x(t), t)\,\delta x + \delta\eta(t) \qquad \left(p(x(t), t) = \left(\frac{\partial f}{\partial x}\right)_{x=x(t)} \right) \tag{4.20}$$

Let us verify that the resolving vector $h(T, t)$ of the system (4.20) is nonsingular.

In fact, in the case under consideration the vector q has the form

$$q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

while the vector Qq has the form

$$Qq = \begin{pmatrix} 0 & 1 \\ -\partial f / \partial x & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

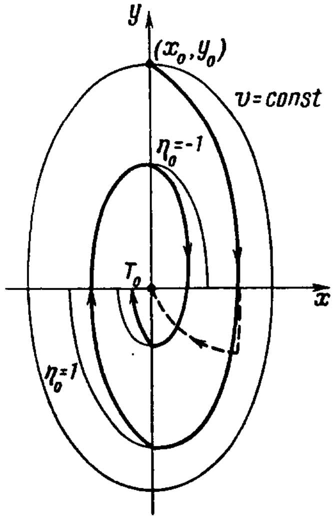
and, consequently, the vectors are not colinear.

The trajectories of the system (4.19) for $\eta = 0$ have the form of spirals described about the origin of the coordinates $x = 0$, $y = 0$. According to the nature of the function $f(x, t)$ these spirals may be either periodic curves, or they may spiral towards the origin or away from the origin as $t \to \infty$. In the particular case when $f(x)$ does not depend on the time t, all solutions of (4.19) are periodic and are given by the level curves $v =$ const. of the function

$$v(x, y) = y^2 + 2\int_0^x f(\xi)\, d\xi$$

Let the function η_0 be determined as follows: $\eta_0 = -1$ if the trajectory passes through the domain $x > 0$, $y > 0$; $\eta_0 = 1$ in the domain $x < 0$, $y < 0$ and $\eta_0 = 0$ for all other x and y. Then for the initial values x_0, y_0, lying sufficiently close to the point $x = 0$, $y = 0$ (in the stationary case for all x_0, y_0) the trajectories (4.19) have the form of spirals which asymptotically approach the point $x = 0$, $y = 0$ as $t \to \infty$ (see figure).

If the time length T_0 of such a trajectory is chosen sufficiently large, the point $x(x_0, y_0, T_0, \eta_0)$, $y(x_0, y_0, T_0, \eta_0)$ will be sufficiently close to the point $x = 0$, $y = 0$, and it is possible to indicate variations $\delta\eta$ of the function $\eta_0(t)$ such that the point $x(x_0, y_0, T_0, \eta_0 \pm \delta\eta)$, $y(x_0, y_0, T_0, \eta_0 + \delta\eta)$ will reach the point $x = 0$, $y = 0$. We will not verify this here since it can be done by methods similar to those considered in Section 2. Consequently, there exists a domain G which embraces the origin of the coordinates $x = 0$, $y = 0$ and is such that for each point x_0, y_0 of this domain there exists a control function $\eta(t)$ which takes the trajectory $x(x_0, y_0, t, \eta)$, $y(x_0, y_0, t\ \eta)$ into the point $x = 0$, $y = 0$. In particular, in the case of a stationary function $f(x)$ such a domain will be the whole plane. Now, according to the results of Sections 2-4, it can be asserted that for every point (x_0, y_0) of G there exists an



optimum trajectory $x(t, \eta^0)$, $y(t, \eta^0)$, connecting the points $x = x_0$, $y = y_0$ and $x = 0$, $y = 0$, and the corresponding control function $\eta^0(t)$ being a piecewise smooth function of the form $\eta^0(t) = \text{sign } (h^0(T^0, t) . l^0)$.

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